

[This question paper contains 4 printed pages.]

(10)

Your Roll No. 2022

Sr. No. of Question Paper : 1132

A

Unique Paper Code : 32351201

Name of the Paper : BMATH203 – Real Analysis

Name of the Course : B.Sc. (H) Mathematics

Semester : II

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All questions are compulsory.
3. Attempt any two parts from each question.
4. All questions carry equal marks.

1. (a) Let  $S$  be a non- empty bounded set of  $\mathbb{R}$ . Let  $b < 0$  and let

$bS = \{bs \mid s \in S\}$ . Prove that  $\inf(bS) = b \sup S$  and  $\sup(bS) = b \inf S$ .

(b) If  $y$  is a positive real number, show that there exists  $n_y \in \mathbb{N}$  such that

$$n_y - 1 \leq y < n_y$$

(c) Let  $X$  be a non- empty set. Let  $f$  and  $g$  be defined on  $\mathbb{R}$  and have bounded ranges in  $\mathbb{R}$ . Show that

$$\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}.$$

P.T.O.

(d) Define a sequence  $\langle e_n \rangle$  by  $e_n = \left(1 + \frac{1}{n}\right)^n$ ,  $\forall n \in \mathbb{N}$ .

Show that  $\langle e_n \rangle$  is bounded and increasing and hence converges. Also, show that  $\lim \langle e_n \rangle$  lies between 2 and 3.

2. (a) State and prove Density theorem.

(b) Let  $A$  and  $B$  be bounded non-empty subsets of  $\mathbb{R}$  and let  $A + B = \{a + b \mid a \in A, b \in B\}$ . Prove that  $\sup(A + B) = \sup A + \sup B$  and  $\inf(A + B) = \inf A + \inf B$

(c) Let  $I_n = \left[0, \frac{1}{n}\right]$ ,  $n \in \mathbb{N}$ . Show that  $\{I_n, n \in \mathbb{N}\}$  is a nested sequence of intervals and  $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$ .

(d) Examine the convergence of the series  $\sum_{n=1}^{\infty} n e^{-n^2}$

3. (a) State and prove Monotone Convergence Theorem.

(b) Let  $(x_n)$  be a sequence of positive real numbers such that  $\lim_{n \rightarrow \infty} (x_n^{1/n}) = L$  exists. Prove that if  $L < 1$ , then  $(x_n)$  converges and  $\lim_{n \rightarrow \infty} (x_n) = 0$ .

(c) Prove that  $\lim_{n \rightarrow \infty} (n^{1/n}) = 1$ .

(d) Use the definition of the limit to show that  $\lim_{n \rightarrow \infty} (x_n) = 0$ , where

$x_n = 1 / \ln(n + 1)$ , for  $n \in \mathbb{N}$ . Also find  $K \in \mathbb{N}$  for  $\varepsilon = \frac{1}{10}$  such that

$|x_n - 0| < \varepsilon$ ,  $\forall n \geq K$ .

4. (a) Let  $X = (x_n)$  and  $Y = (y_n)$  be sequences of real numbers that converge to  $x$  and  $y$  respectively and if  $y \neq 0$ . Then the quotient sequence  $X/Y$  converges to  $x/y$ .

(b) State and prove Squeeze Theorem. Also find  $\lim_{n \rightarrow \infty} \left(\frac{\sin n}{n}\right)$

(c) State Cauchy Convergence Criterion for Sequences.

Let  $X = (x_n)$  be defined by  $x_1 = 1, x_2 = 2$  and  $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$

for  $n > 2$ . Prove that the sequence  $X$  is convergent.

(d) Discuss the convergence of the sequence  $(x_n)$ , where

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}, \text{ for each } n \in \mathbb{N}.$$

5. (a) Suppose the  $k$ th partial sum of  $\sum_{n=1}^{\infty} x_n$  is  $s_k = \frac{k}{k+1}$ . Find the corresponding series and general term  $x_n$ . Prove that the series converges and then find the limit.

(b) Prove that the harmonic series  $\sum \frac{1}{n}$  diverges (despite the fact that  $\lim \frac{1}{n} = 0$ ).

(c) Test for convergence, the following series:

$$(i) \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \quad (ii) \frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \dots$$

(d) Show that the series  $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots, p > 0$  converges absolutely for  $p > 1$  and conditionally for  $0 < p \leq 1$ .

6. (a) Prove that if  $\sum_{n=1}^{\infty} a_n$  is a series of positive terms and that its partial sums are bounded, then  $\sum_{n=1}^{\infty} a_n$  converges. Show that this is not necessarily true if  $\sum_{n=1}^{\infty} a_n$  is not a series of positive terms.

(b) State and prove the limit comparison test.

(c) Test for convergence, the following series:

$$(i) 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots$$

$$(ii) \sum_{n=1}^{\infty} 3^{-n-(-1)^n}$$

- (d) Define absolute and conditional convergence of an alternating series. Show that the series  $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$  is conditionally convergent but not absolutely.

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